Numerical Analysis of Rotationally Symmetric Shells under Transient Loadings

Troy Alvin Smith*
U. S. Army Missile Command, Redstone Arsenal, Ala.

This report involves the development of procedures for determining the dynamic response of rotationally symmetric open ended thin shells of revolution under time dependent distributed impulsive and thermal loadings. The solution is restricted to thermal loadings which are small in the circumferential direction but which may vary in any manner in the meridional direction of the shell. Inertia forces are considered in a direction normal to the middle surface and in a direction along the meridians of the shell. Time dependent boundary conditions may be prescribed at each of the two edges of the shell. The field equations are derived in the form of eight first-order partial differential equations with respect to the meridional direction of the shell. The solution for each Fourier harmonic is obtained by employing finite difference representations for all time and spatial derivatives. The complete system of equations is solved implicitly for the first time increment, whereas explicit relations are used for the meridional and transverse displacements for the second and succeeding time increments. The developed equations have been programmed in FORTRAN IV language for solution by computer. Solutions obtained with this program for typical shells and loadings are found to be stable and in agreement for a range of values of the space and time increments. It is concluded that the finite difference methods employed here constitute a most expeditious procedure for obtaining reliable and accurate values of the response histories of rotationally symmetric shells of revolution under transient loadings.

Nomenclature

D = $Eh^3/12(1 - \nu^2)$, flexural rigidity of shell
D° = $D \times 10^{-6}$
E = Young's modulus
E° = $E \times 10^{-6}$
g = acceleration constant
H = $(1/R_\phi) - (\sin\phi/r)$
H_1 = $H/\sin\phi$
h = thickness of shell
J = $(1/R_\phi) + (\sin\phi/r)$
J_1 = $J/\sin\phi$
K = $Eh/(1 - \nu^2)$, extensional rigidity of shell
K° = $K \times 10^{-6}$
L = $1/[1 + (D \sin^2\phi/Kr^2)]$
L_1 = $L/\sin\phi$
$M_\theta, M_\phi, M_{\theta\phi}$ = moment stress resultants
m_ϕ = moment of mechanical surface loads
N, Q = effective shear resultants
$N_\theta, N_\phi, N_{\theta\phi}$ = membrane stress resultants
n = integer, designating nth Fourier component
p, p_ϕ = components of mechanical surface loads
Q_θ, Q_ϕ = transverse shear resultants
R_θ, R_ϕ = principal radii of curvature of middle surface of shell
r = distance of point on middle surface of shell from axis of symmetry
s = distance from an arbitrary origin along meridian in positive direction of ϕ
T, T_0, T_1 = temperature increment and temperature resultants
t = independent time variable
Δt = increment of the time variable t
U = $(1/R_\phi) + (\nu \sin\phi/r)$
U_1 = $U/\sin\phi$
u_θ, u_ϕ, w = components of displacement of middle surface of shell
u_θ^0 = $u_\theta \times 10^6$
u_ϕ^0 = $u_\phi \times 10^6$
w^0 = $w \times 10^6$
z = distance of point on middle surface of shell measured from origin along axis of symmetry
Δz = increment of the space variable z
α = coefficient of thermal expansion of shell material
β_θ, β_ϕ = angles of rotation of normal to middle surface of shell
β_ϕ^0 = $\beta_\phi \times 10^6$
γ = weight of shell material per unit volume
θ, ϕ, ρ = coordinates of any point of shell
ν = Poisson's ratio

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* Aerospace Engineer.

Introduction

THE determination of the dynamic response of a general shell of revolution subjected to time dependent distributed surface and thermal loadings and with arbitrary time dependent boundary conditions constitutes a rather involved assignment. No known closed form solutions for the general shell even in the absence of thermal loadings were found during a survey of the literature on the theory and analysis of shells. Indeed, no known closed form solutions were found for the general shell even under static loads. However, several investigators have successfully solved the static problem of a general shell by numerical methods.

These investigators include Penny,[1] who solved the symmetric bending problem of a general shell in 1961 by finite differences; Radkowski, Davis, and Bolduc,[2] who solved the axisymmetric static problem in 1962 by finite differences;

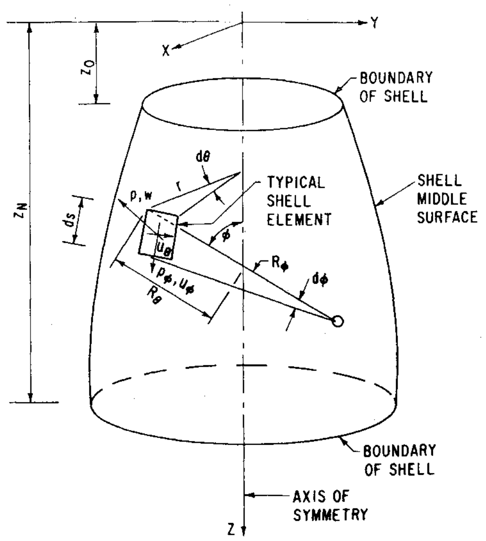

Fig. 1 Typical shell of revolution.

and Budiansky and Radkowski,[3] who employed finite difference methods to solve the unsymmetrical static bending problem in 1963. The solution to the static problem of rotationally symmetric shells of revolution subjected to both symmetrical and nonsymmetrical loading was obtained also by Kalnins[4] in 1964. Starting with the equations of the linear classical bending theory of shells, in which the thermal effects were included, he derived a system of eight first-order ordinary differential equations which he solved by direct numerical integration over preselected segments of the shell. Gaussian elimination was used to solve the resulting system of matrix equations obtained by providing continuity of the fundamental variables at the segmental division points. In 1965, Percy, Pian, Klein, and Navaratna[5] also developed a finite element technique for the analysis of shells of revolution under both axisymmetric and asymmetric static loading by idealizing the shell as a series of conical frusta.

The solution for the free vibration characteristics of rotationally symmetric shells with meridional variations in the shell parameters by means of his multisegment direct numerical integration approach was also obtained by Kalnins[6] in 1964. Subsequently, in 1965, the solution for the response of an arbitrary shell subjected to time dependent surface loadings was obtained by Kraus and Kalnins[7] by means of the classical method of spectral representation. The solution was expanded in terms of the modes of free vibration as determined previously by Kalnins,[6] and the orthogonality of the normal modes was proved for an arbitrary shell.

In 1966 Klein[8] also published an article in which he describes a matrix displacement finite element approach to the linear elastic analysis of multilayer shells of revolution under axisymmetric and asymmetric dynamic and impulsive loadings. The method of solution involves the idealization of the shell as a series of conical frusta joined at nodal circles.

The solution for the dynamic response of a circular cylindrical shell with constant geometric and material properties and under isothermal conditions was also obtained in 1966 by Johnson and Greif[9] for the case of linear elastic shell response. These authors represented the field equations in the form of four second-order partial differential equations with respect to the meridional direction of the cylinder and obtained solutions for each Fourier harmonic by employing finite difference representations for both the time and the meridional coordinate derivatives. They obtained and compared solutions by both the implicit and the explicit methods.

None of the aforementioned formulations, however, contains all of the features necessary or desirable for the solution of the general dynamic shell problem. In particular, the desirability of being able to impose arbitrary time dependent boundary conditions with maximum facility at each edge of the shell, the time dependency of the thermal loadings along the meridian of the shell, and the need of a capability to analyze these general shells for rapid time fluctuations in the loading dictate that a general and more flexible method of analysis be utilized. This general and more flexible method of analysis involves the finite difference methods to be discussed in the present report.

Governing Differential Equations

The development of our system of governing equations will be based upon the linear classical theory of shells as given by Reissner.[10] In the development of our equations, only surface loadings and inertia forces normal to the middle surface and along the meridians of the shell will be considered. Surface loadings and inertia forces in the circumferential direction of the shell and all rotary inertia terms will be neglected.

The geometry and coordinate system for the middle surface of our shell is shown in Fig. 1. Shell element membrane and shear forces are shown in Fig. 2, and shell element bending and twisting moments are shown in Fig. 3.

The position of any point on the middle surface of the shell may be defined for convenience in terms of the three independent coordinates θ, z, and t. The principal radii of curvature of the shell may be expressed as

$$R_\phi = -[1 + (r_{,z})^2]^{3/2}/r_{,zz} \tag{1a}$$

$$R_\theta = r[1 + (r_{,z})^2]^{1/2} \tag{1b}$$

To account for variation of temperature through the thickness, it is convenient to introduce temperature resultants by integrating the temperature distribution through the thickness. Following Kalnins,[4] these resultants are

$$T_0(\theta,z,t) = (1/h)\int_{-h/2}^{h/2} T(\theta,z,\rho,t)d\rho \tag{2a}$$

$$T_1(\theta,z,t) = (12/h^3)\int_{-h/2}^{h/2} \rho T(\theta,z,\rho,t)d\rho \tag{2b}$$

In defining the material properties of the shell mathematically, we consider only the effect of the thermal loadings on the values of the quantities E, ν, and α. These quantities depend upon the temperature increment T. Since $T = T(\theta,z,\rho,t)$, it is convenient to base the values of E, ν, and α upon the temperature resultant T_0. If $T_0 = T_0(\theta,z,t)$, we cannot obtain uncoupled solutions to our equations when the effect of the thermal loadings on the quantities E, ν, and α is considered. For thermal loadings which are axisymmetric, $T_0 = T_0\ (z,t)$, and our equations become uncoupled for the separate Fourier components of loading. We thus have two separate cases of thermal loadings to consider. For the first

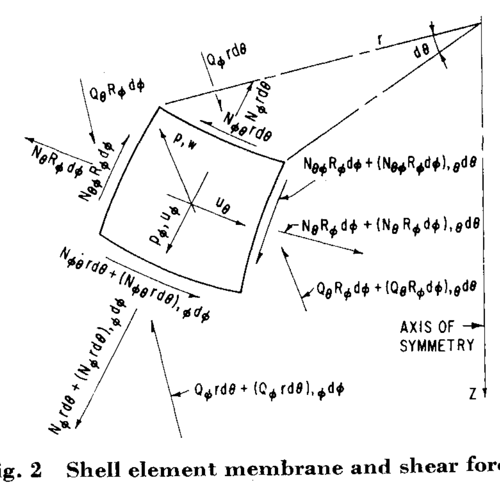

Fig. 2 Shell element membrane and shear forces.

case, in which $T_0 = T_0(\theta,z,t)$, we neglect the effect of the thermal loadings on E, ν, and α and assume that

$$E = \text{const} \quad \nu = \text{const} \quad \alpha = \text{const} \tag{3}$$

For the second case, in which $T_0 = T_0\ (z,t)$, we consider the effect of thermal loadings on E, ν, and α and have

$$E = E(z,t) \quad \nu = \nu(z,t) \quad \alpha = \alpha(z,t) \tag{4}$$

The field equations are of order 8 with respect to the coordinate ϕ. Kalnins[4] has shown that this system may be reduced to a system of eight first-order equations in the eight unknown quantities which enter into the natural boundary conditions at ϕ = const. In the classical theory of shells, the quantities which appear in the natural boundary conditions on a rotationally symmetric edge of a shell of revolution are the generalized displacements u_ϕ, u_θ, w, and β_ϕ and the generalized forces N_ϕ, M_ϕ, N, and Q. The quantities N and Q are the effective shear resultants and are defined as

$$N = N_{\theta\phi} + (\sin\phi/r)M_{\theta\phi} \tag{5a}$$

$$Q = Q_\phi + (1/r)M_{\theta\phi,\theta} \tag{5b}$$

The system of equations obtained by following the procedure used by Kalnins[4] involves the three independent variables θ, z, and t. To reduce the system to the two independent variables z and t, we express all loadings and dependent variables in the circumferential direction of the shell in the form of Fourier series expansions. We will truncate these infinite series at a finite number of terms for the solution of specific shell problems.

The Fourier series representations of the loadings are

$$p_\phi = \sum_{n=0}^{P} p_{\phi n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{p}_{\phi n}\ (z,t)\sin n\theta \tag{6}$$

$$m_\phi = \sum_{n=0}^{P} m_{\phi n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{m}_{\phi n}\ (z,t)\sin n\theta \tag{7}$$

$$p = \sum_{n=0}^{P} p_{n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{p}_{n}\ (z,t)\sin n\theta \tag{8}$$

$$T_0 = \sum_{n=0}^{P} T_{0n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{T}_{0n}\ (z,t)\sin n\theta \tag{9}$$

$$T_1 = \sum_{n=0}^{P} T_{1n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{T}_{1n}\ (z,t)\sin n\theta \tag{10}$$

The Fourier series representations of the fundamental variables in our governing equations are

$$w = \sum_{n=0}^{P} w_{n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{w}_{n}\ (z,t)\sin n\theta \tag{11}$$

$$u_\phi = \sum_{n=0}^{P} u_{\phi n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{u}_{\phi n}\ (z,t)\sin n\theta \tag{12}$$

$$\beta_\phi = \sum_{n=0}^{P} \beta_{\phi n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{\beta}_{\phi n}\ (z,t)\sin n\theta \tag{13}$$

$$N_\phi = \sum_{n=0}^{P} N_{\phi n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{N}_{\phi n}\ (z,t)\sin n\theta \tag{14}$$

$$M_\phi = \sum_{n=0}^{P} M_{\phi n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{M}_{\phi n}\ (z,t)\sin n\theta \tag{15}$$

$$Q = \sum_{n=0}^{P} Q_{n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{Q}_{n}\ (z,t)\sin n\theta \tag{16}$$

$$u_\theta = \sum_{n=1}^{P} u_{\theta n}\ (z,t)\sin n\theta + \sum_{n=0}^{P} \bar{u}_{\theta n}\ (z,t)\cos n\theta \tag{17}$$

$$N = \sum_{n=1}^{P} N_{n}\ (z,t)\sin n\theta + \sum_{n=0}^{P} \bar{N}_{n}\ (z,t)\cos n\theta \tag{18}$$

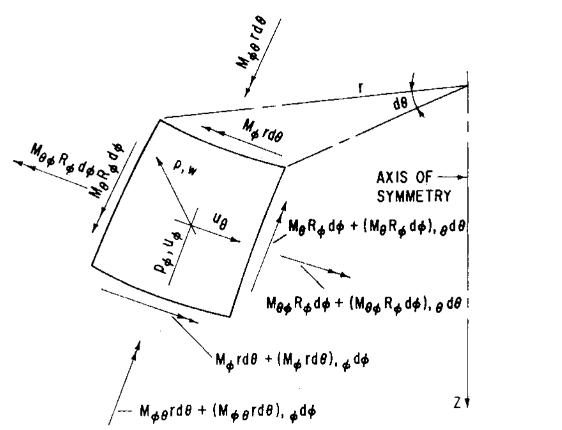

Fig. 3 Shell element bending and twisting moments.

The stress resultants other than the fundamental variables will be represented as

$$N_\theta = \sum_{n=0}^{P} N_{\theta n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{N}_{\theta n}\ (z,t)\sin n\theta \tag{19}$$

$$M_\theta = \sum_{n=0}^{P} M_{\theta n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{M}_{\theta n}\ (z,t)\sin n\theta \tag{20}$$

$$Q_\phi = \sum_{n=0}^{P} Q_{\phi n}\ (z,t)\cos n\theta + \sum_{n=1}^{P} \bar{Q}_{\phi n}\ (z,t)\sin n\theta \tag{21}$$

$$N_{\theta\phi} = \sum_{n=1}^{P} N_{\theta\phi n}\ (z,t)\sin n\theta + \sum_{n=0}^{P} \bar{N}_{\theta\phi n}\ (z,t)\cos n\theta \tag{22}$$

$$M_{\theta\phi} = \sum_{n=1}^{P} M_{\theta\phi n}\ (z,t)\sin n\theta + \sum_{n=0}^{P} \bar{M}_{\theta\phi n}\ (z,t)\cos n\theta \tag{23}$$

$$Q_\theta = \sum_{n=1}^{P} Q_{\theta n}\ (z,t)\sin n\theta + \sum_{n=0}^{P} \bar{Q}_{\theta n}\ (z,t)\cos n\theta \tag{24}$$

With this representation of the loadings and variables, we obtain $P + 1$ separate decoupled systems of equations in the two independent variables z and t. The details of the derivation are given in Ref. 11 but are omitted here for the sake of brevity. In developing our final equations we have deleted certain terms which should not appear if we maintain consistency with our previous assumption which entered into the development of the strain-displacement relations, namely that the shell is sufficiently thin that the quantity $1 + h^2/12r^2$ can be taken to be equal to unity. We note also that we obtain two separate sets of equations, one set for the variables which are not designated with a bar and another set for the variables which are designated with a bar. Here and elsewhere in the sequel, where double signs occur in the equations, the upper sign is to accompany the first set of equations and the lower sign is to apply to the second set. Single signs will of course apply to both sets. The desired field equations for each Fourier harmonic are

$$w_{n,z} = (1/\sin\phi)[(1/R_\phi)u_{\phi n} - \beta_{\phi n}] \tag{25}$$

$$u_{\phi n,z} = U_1 w_n - (\nu\cot\phi/r)u_{\phi n} \mp (\nu n/r\sin\phi)u_{\theta n} + (1/K\sin\phi)N_{\phi n} + (\alpha/\sin\phi)(1+\nu)T_{0n} \tag{26}$$

$$u_{\theta n,z} = \pm(2nD\cos\phi/Kr^3)w_n \pm (n/r\sin\phi)u_{\phi n} + (\cot\phi/r)u_{\theta n} \pm (2nD/Kr^2)\beta_{\phi n} + [2/(1-\nu)K\sin\phi]N_n \tag{27}$$

$$\beta_{\phi n,z} = -(\nu n^2/r^2\sin\phi)w_n \mp (\nu n/r^2)u_{\theta n} - (\nu\cot\phi/r)\beta_{\phi n} + (1/D\sin\phi)M_{\phi n} + (\alpha/\sin\phi)(1+\nu)T_{1n} \tag{28}$$

$$N_{n,z} = \pm(n/r^2)(1-\nu)[(1+\nu)(n^2D/r^2) + (1+\nu)K]w_n \pm (1-\nu^2)(nK\cot\phi/r^2)u_{\phi n} + (1-\nu^2)(n^2K/r^2\sin\phi)u_{\theta n} \pm (nD/r^2)\times (1-\nu)\cos\phi[(1+\nu)/r - H_1]\beta_{\phi n} \pm (\nu n/r\sin\phi)N_{\phi n} - (2\cot\phi/r)N_n \pm (\nu n/r^2)M_{\phi n} \mp (1-\nu^2)(n\alpha/r)\times [(K/\sin\phi)T_{0n} + (D/r)T_{1n}] \quad (29)$$

$$Q_{n,z} = (1-\nu)(1/r^4\sin\phi)[(1+\nu)n^4D + 2n^2D\cos^2\phi + (1+\nu)Kr^2\sin^2\phi]w_n + (1-\nu)(1/r^2)\cos\phi[(1+\nu)K - (n^2/r)DJ_1]u_{\phi n} \pm (1-\nu)(n/r^2)[(1+\nu)\times (n^2D/r^2) + (1+\nu)K]u_{\theta n} + n^2D(1-\nu)\times (3+\nu)(\cot\phi/r^3)\beta_{\phi n} - (\cot\phi/r)Q_n + U_1N_{\phi n} + (\nu n^2/r^2\sin\phi)M_{\phi n} \mp (2nD\cos\phi/Kr^3)N_n - (1/\sin\phi)p_n - (1-\nu^2)(\alpha/r)[KT_{0n} + (n^2D/r\sin\phi)T_{1n}] + (\gamma h/g\sin\phi)w_{n,tt} \quad (30)$$

$$N_{\phi n,z} = (1-\nu)(\cos\phi/r^2)[(1+\nu)K - (n^2/r)DJ_1]w_n + (1-\nu)(1/r^2)[(1+\nu)K\times \cos\phi\cot\phi + (n^2D/2)J_1^2\sin\phi]u_{\phi n} \pm (1-\nu^2)(nK\cot\phi/r^2)u_{\theta n} - (n^2/r^2)(1-\nu)DJ_1\beta_{\phi n} - (1/R_\phi\sin\phi)Q_n - (1-\nu)(\cot\phi/r)N_{\phi n} \mp (n/r\sin\phi)N_n - (1/\sin\phi)p_{\phi n} - (1-\nu^2)(\alpha K\cot\phi/r)T_{0n} + (\gamma h/g\sin\phi)u_{\phi n,tt} \quad (31)$$

$$M_{\phi n,z} = (n^2D\cot\phi/r^3)(1-\nu)(3+\nu)w_n - (n^2DJ_1/r^2)(1-\nu)u_{\phi n} \pm (1-\nu)(nD\cos\phi/r^2)\times [(1+\nu)(1/r) - H_1]u_{\theta n} + (1-\nu)\times (D/r^2\sin\phi)[(1+\nu)\cos^2\phi + 2n^2]\beta_{\phi n} + (1/\sin\phi)Q_n \mp (2nD/Kr^2)N_n - (1-\nu)(\cot\phi/r)M_{\phi n} - (1/\sin\phi)m_{\phi n} - (1-\nu^2)(\alpha D\cot\phi/r)T_{1n} \quad (32)$$

The equations for the Fourier components of the stress resultants other than the fundamental variables are

$$N_{\theta n} = \nu N_{\phi n} + (1-\nu^2)(K/r)(\pm nu_{\theta n} + u_{\phi n}\cos\phi + w_n\sin\phi) - (1-\nu^2)\alpha KT_{0n} \quad (33)$$

$$M_{\theta n} = \nu M_{\phi n} + (1-\nu^2)(D/r)[(n^2/r)w_n \pm (n/r)\sin\phi u_{\theta n} + \beta_{\phi n}\cos\phi] - (1-\nu^2)\alpha DT_{1n} \quad (34)$$

$$M_{\theta\phi n} = (1-\nu)(D/2r)[\mp 2n\beta_{\phi n} \mp (2n\cos\phi/r)w_n + H\cos\phi u_{\theta n} \pm nJu_{\phi n}] + (D\sin\phi/Kr)N_n \quad (35)$$

$$Q_{\theta n} = \mp(n/r)M_{\theta n} + M_{\theta\phi n,z}\sin\phi + (2\cos\phi/r)M_{\theta\phi n} \quad (36)$$

$$N_{\theta\phi n} = N_n - (\sin\phi/r)M_{\theta\phi n} \quad (37)$$

$$Q_{\phi n} = Q_n \mp (n/r)M_{\theta\phi n} \quad (38)$$

To effect a finite difference solution of our system of equations, it will be convenient to have the generalized forces expressed in terms of the generalized displacements and loadings. After expanding all of the dependent variables and loadings in Fourier series and deleting derivatives of material parameters and T_1 with respect to the coordinate z from our equation for Q, we obtain

$$N_n = (1-\nu)(\cos\phi/2r)[(D\sin\phi/r)(H - \sin\phi/r) - K]u_{\theta n} + (1-\nu)(\sin\phi/2)\times [(D\sin^2\phi/r^2) + K]u_{\theta n,z} \mp (1-\nu)(nK/2r)u_{\phi n} \mp (1-\nu)(nD\sin\phi/2r^2)\beta_{\phi n} \mp (1-\nu)(nD\sin\phi\cos\phi/r^3)w_n \pm (1-\nu)(nD\sin^2\phi/2r^2)w_{n,z} \quad (39)$$

$$Q_n = -(1-\nu)(n^2D/r^2)\beta_{\phi n} - (3-\nu)\times (n^2D\cos\phi/r^3)w_n \pm (nD\cos\phi/r^2)[(1/R_\phi) - (3-\nu)(\sin\phi/r)]u_{\theta n} + (n^2D\sin\phi/r^2)w_{n,z} \pm (nD\sin^2\phi/r^2)u_{\theta n,z} + D\sin^2\phi\beta_{\phi n,zz} + DJ\cos\phi\beta_{\phi n,z} - (D/r)[(\nu\sin\phi/R_\phi) + (\cos^2\phi/r)]\beta_{\phi n} + m_{\phi n} \quad (40)$$

$$N_{\phi n} = K[\sin\phi u_{\phi n,z} + (w_n/R_\phi) \pm (\nu n/r)u_{\theta n} + (\nu/r)(u_{\phi n}\cos\phi + w_n\sin\phi)] - \alpha(1+\nu)KT_{0n} \quad (41)$$

$$M_{\phi n} = D[\sin\phi\beta_{\phi n,z} + (\nu n^2/r^2)w_n \pm (\nu n\sin\phi/r^2)u_{\theta n} + (\nu\cos\phi/r)\beta_{\phi n}] - \alpha(1+\nu)DT_{1n} \quad (42)$$

Conversion of Equations to Finite Difference Form

The analysis of our shell of revolution now consists of the solution of the system of Eqs. (25–32) subject to the initial conditions and to the boundary conditions for each Fourier component of loading and a summation of the results to obtain the values of the eight fundamental variables. Values of the stress resultants other than the fundamental variables may then be obtained by summing the results for the Fourier components given by Eqs. (33–38).

To solve the aforementioned system of equations for each Fourier component of loading we replace all unknown derivatives in the equations by their finite difference equivalents to obtain a system of algebraic equations which may be applied at successive increments of the time variable. If the shell meridian is divided into N increments, we obtain $8(N-1)$ algebraic equations in the coordinate z by writing Eqs. (25–32) at each of the $N-1$ interior points on the shell meridian.

We have four additional equations in the coordinate z involving the four prescribed boundary values at each of the boundaries z_0 and z_N. Thus, we have a total of $8N$ algebraic equations and must supplement them with eight additional independent equations to effect a solution for the $8(N+1)$ unknown quantities. We choose for the eight additional equations the relations given by Eqs. (39–42) evaluated at each of the boundaries z_0 and z_N.

By representing the initial velocities by finite central differences and by representing the accelerations at time t_1 by finite backward differences, we obtain the final expressions for the accelerations at time t_1 to be given typically by

$$\ddot{w}_n(z,t_1) = \frac{2[w_n(z,t_1) - w_n(z,t_0) - (\Delta t)\dot{w}_n(z,t_0)]}{(\Delta t)^2} \quad (43)$$

For times $t \geq t_0 + 2\Delta t$, we represent the accelerations in Eqs. (30) and (31) more accurately by finite central differences about $t - \Delta t$. The expressions for the accelerations at time $t - \Delta t$ are typically

$$\ddot{w}_n(z,t-\Delta t) = \frac{w_n(z,t-2\Delta t) - 2w_n(z,t-\Delta t) + w_n(z,t)}{(\Delta t)^2} \quad (44)$$

The first derivatives which appear in Eqs. (25–32) and in Eq. (36) when applied within the boundary edges of the shell will be represented typically as

$$\beta_{\phi n,z} = [\beta_{\phi n}(z+\Delta z) - \beta_{\phi n}(z-\Delta z)]/2\Delta z \quad (45)$$

The first derivatives at the boundaries of the shell in Eqs. (39–42) and in Eq. (36) will be represented typically as

$$\beta_{\phi n,z}(z_0) = (1/6\Delta z)[-11\beta_{\phi n}(z_0) + 18\beta_{\phi n}(z_1) - 9\beta_{\phi n}(z_2) + 2\beta_{\phi n}(z_3)]$$

$$\beta_{\phi n,z}(z_N) = (1/6\Delta z)[11\beta_{\phi n}(z_N) - 18\beta_{\phi n}(z_{N-1}) + 9\beta_{\phi n}(z_{N-2}) - 2\beta_{\phi n}(z_{N-3})] \quad (46)$$

The second derivatives at the boundaries of the shell in

Eq. (40) will be represented as

$$\beta_{\phi n,zz}(z_0) = [1/(\Delta z)^2][2\beta_{\phi n}(z_0) - 5\beta_{\phi n}(z_1) + 4\beta_{\phi n}(z_2) - \beta_{\phi n}(z_3)]$$

$$\beta_{\phi n,zz}(z_N) = [1/(\Delta z)^2][2\beta_{\phi n}(z_N) - 5\beta_{\phi n}(z_{N-1}) + 4\beta_{\phi n}(z_{N-2}) - \beta_{\phi n}(z_{N-3})] \quad (47)$$

In order to produce more nearly equal coefficients of the eight dependent variables in our system of equations, we also define a new set of displacement variables as follows:

$$w_n(z,t) = w_n{}^0(z,t) \times 10^{-6} \quad (48a)$$

$$u_{\phi n}(z,t) = u_{\phi n}{}^0(z,t) \times 10^{-6} \quad (48b)$$

$$u_{\theta n}(z,t) = u_{\theta n}{}^0(z,t) \times 10^{-6} \quad (48c)$$

$$\beta_{\phi n}(z,t) = \beta_{\phi n}{}^0(z,t) \times 10^{-6} \quad (48d)$$

We define new quantities E^0, D^0, and K^0 as follows:

$$E^0 = E \times 10^{-6} \quad (49a)$$

$$D^0 = D \times 10^{-6} \quad (49b)$$

$$K^0 = K \times 10^{-6} \quad (49c)$$

We convert our field equations to finite difference form for the first time increment by substituting Eqs. (43) and (45) into the system of equations (25–32) and using Eqs. (48) and (49). We express Eqs. (39–42) in finite difference form for the boundaries z_0 and z_N by substituting the appropriate derivatives given in the form of Eqs. (46) and (47) into Eqs. (39–42). Together with four equations involving the prescribed boundary values at each boundary of the shell we thus obtain $8(N + 1)$ algebraic equations, the solution of which, when used in conjunction with Eqs. (48) and (49), yields the $8(N + 1)$ unknown quantities for the shell for each Fourier component n at the time t_1.

To convert our field equations to finite difference form for the second and subsequent time increments, we employ the relations defined by Eqs. (48) and (49) and substitute Eqs. (44) and (45) into the system of Eqs. (25–32). From the resulting system of equations we obtain explicit expressions for $w_n(z,t)$ and $u_{\phi n}(z,t)$ on the interval $z_1 \leqq z \leqq z_{N-1}$ and for $\beta_{\phi n}(z,t)$ on the interval $z_2 \leqq z \leqq z_{N-2}$. From the remainder of our system of equations, consisting of Eqs. (26–29) and (32) expressed in finite difference form for each interior point, Eq. (25) written in finite difference form at z_1 and z_{N-1}, four equations involving the prescribed boundary values at each boundary, and Eqs. (39–42) evaluated at each boundary in finite difference form after using Eqs. (48) and (49), we obtain $[5(N - 1) + 2] + 16$ simultaneous algebraic equations. The solution of this latter group of equations, when used in conjunction with Eqs. (48) and (49), yields the remaining $[5(N - 1) + 2] + 16$ unknown quantities for the shell for each Fourier component n at time t.

The algebraic relations for the Fourier components of N_θ, M_θ, $M_{\theta\phi}$, $N_{\theta\phi}$, and Q_ϕ are given respectively by Eqs. (33–35, 37, and 38). We obtain the finite difference relations for $Q_{\theta n}$ by substituting the appropriate derivatives given in the forms of Eqs. (45) and (46) into Eq. (36). The governing finite difference equations described herein for both t_1 and t are given in Ref. (11) but are omitted here for the sake of brevity.

Selection of Meridional and Time Increments for Solution of Finite Difference Equations

In the solution of our system of finite difference equations, choices must be made for the increments Δz and Δt. For the solution of the finite difference equations to converge to the true solution of the differential equations which they represent, it is necessary that these increments be chosen so that the solution of the finite difference equations is stable.

Methods for determining necessary and sufficient conditions for stability of solution are well established for systems of partial differential equations with constant coefficients and simple periodic boundary conditions. However, the development of tractable procedures for determining the stability limits for general systems of finite difference equations is in a relative state of infancy. Thus, in the present analysis, we will determine by trial for each shell to be analyzed appropriate choices of increments Δt which may be used with a given increment Δz to obtain stable solutions. For any given increment Δz we expect a limited range of values of Δt for which stable solutions will be obtained.

The selection of an increment Δz will be based on the requirement that the finite difference solution for the static problem must converge to the true solution of the differential equations. We thus need only to choose the increment Δz to minimize truncation and possibly round-off errors. We expect round-off errors to be significant as Δz approaches zero, and truncation errors will be significant if Δz is chosen to be too large. Upon the basis of static solutions obtained for typical shells, it appears that sufficiently accurate solutions are usually obtained if the increment Δz is chosen to be from one to two times the thickness h of the shell. Furthermore, the choice of an increment Δz within this range will take account of the exciting of vibration modes of relatively short wave lengths for the dynamic shell problem. We will therefore choose the increment Δz to lie within this range for the solution of our finite difference equations for the dynamic shell problem.

In the choice of a time increment Δt, the following three considerations are necessary. a) The increment must be sufficiently small to closely represent the loadings. b) The increment must be a small fraction of the period of the highest significant vibration mode which is excited by the loading in order that the history of the shell response may be adequately tracked. c) The increment must be of the proper magnitude to produce, in conjunction with the chosen meridional increment Δz, a stable solution to the system of finite difference equations.

In satisfying these three considerations, we wish to choose the increment Δt to be as large as possible in order to minimize computation time. In regard to the first consideration, we determine a suitable range of values of Δt by a study of the loading versus time functions. In regard to the second consideration, we expect that the significant response of the shell will usually be governed by the first few of the lower modes. We associate with each Fourier component n a family of modes consisting of the fundamental mode and the higher modes. We expect the frequency of the fundamental mode for $n = 0$ to be somewhat higher than the fundamental frequencies for the next few values of n. For some small value of n, however, we obtain a minimum value of the set of fundamental frequencies. For greater values of n, we expect the magnitude of the frequencies to increase with increased n. Thus, in our analysis, we will determine with reasonable accuracy by Rayleigh's method only the fundamental periods of vibration for $n = 0$ and for the highest Fourier component n which we use to represent the loadings and fundamental variables. We will then make our choice of Δt by a direct consideration of only the one of these two Fourier components which has the highest fundamental frequency. We will choose Δt to be some small fraction of this shortest fundamental period with the expectation that the chosen Δt will also be suitable to determine the response due to all other significant modes. Thus, our initial choice of an increment Δt will be the largest Δt considered both to represent the loading versus time functions and to be suitable for determining the response due to the significant vibration modes. In regard to stability of the solution, we will analyze our shell with both our initial choice of Δt and a second choice of Δt which is smaller than the initial choice. If the two solutions agree at corresponding times for the same increment Δz, we accept both solutions to be stable and valid. If the two solutions do not agree, we will choose still smaller values of Δt until two solutions are in agreement.

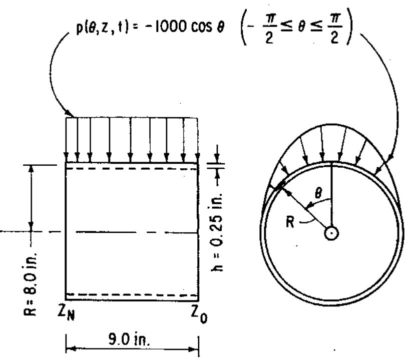

Fig. 4 Typical cylindrical shell and loading.

To determine the frequency by Rayleigh's method, we make a reasonable assumption of the extreme deflected configuration of the shell during free vibration, thus reducing the system to one degree of freedom. The frequency thus determined will be higher than the actual fundamental frequency unless we assume the correct fundamental mode shape, in which case the frequency will be determined precisely. Thus, any Δt selected to be some fraction μ of the calculated fundamental period will be a smaller fraction of the actual fundamental period. This is a desirable condition if the frequency is not found exactly. We assume the extreme deflected position of the shell during free oscillations to be that produced by the Fourier component of loading for which γh is the coefficient acting statically in the directions of the coordinates w_n and $u_{\phi n}$. With w_n, $u_{\phi n}$, and $u_{\theta n}$ used to represent the extreme deflected position of the shell under this assumed loading, we obtain the formulas for the fundamental frequencies to be

Table 1 Illustrative example solutions for $w(z_{18},t)$ at $\theta = 0$ with $\Delta t = 0.125 \times 10^{-5}$ sec and $\Delta t = 0.150 \times 10^{-5}$ sec

t (10^{-5} sec)	$w(z_{18},t)$, in. $\Delta t = 0.125 \times 10^{-5}$ sec	$w(z_{18},t)$, in. $\Delta t = 0.150 \times 10^{-5}$ sec
0	0	0
1.50	-5.9919×10^{-4}	-5.9920×10^{-4}
3.00	-2.3287×10^{-3}	-2.3287×10^{-3}
4.50	-4.9952×10^{-3}	-4.9953×10^{-3}
6.00	-8.3240×10^{-3}	-8.3243×10^{-3}
7.50	-1.1830×10^{-2}	-1.1830×10^{-2}
9.00	-1.4879×10^{-2}	-1.4879×10^{-2}
10.50	-1.7711×10^{-2}	-1.7712×10^{-2}
12.00	-2.0338×10^{-2}	-2.0338×10^{-2}
13.50	-2.2437×10^{-2}	-2.2438×10^{-2}
15.00	-2.3588×10^{-2}	-2.3588×10^{-2}
16.50	-2.3468×10^{-2}	-2.3468×10^{-2}
18.00	-2.1891×10^{-2}	-2.1891×10^{-2}
19.50	-1.8958×10^{-2}	-1.8957×10^{-2}
21.00	-1.4998×10^{-2}	-1.4998×10^{-2}
22.50	-1.0414×10^{-2}	-1.0413×10^{-2}
24.00	-5.5277×10^{-3}	-5.5253×10^{-3}
25.50	-1.3302×10^{-3}	-1.3303×10^{-3}
27.00	1.4768×10^{-3}	1.4770×10^{-3}
28.50	2.9147×10^{-3}	2.9139×10^{-3}
30.00	3.0241×10^{-3}	3.0243×10^{-3}
31.50	2.0248×10^{-3}	2.0241×10^{-3}
33.00	-4.5644×10^{-5}	-4.5408×10^{-5}

$$\omega^2 = \frac{g\left\{(hr/\sin\phi)[w_n(z_0) + u_{\phi n}(z_0)](\Delta z/2) + \sum_{i=1}^{N-1}(hr/\sin\phi)[w_n(z_i) + u_{\phi n}(z_i)]\Delta z + (hr/\sin\phi)[w_n(z_N) + u_{\phi n}(z_N)](\Delta z/2)\right\}}{\left\{(hr/\sin\phi)\overline{[w_n(z_0)]^2 + [u_{\phi n}(z_0)]^2}(\Delta z/2) + \sum_{i=1}^{N-1}(hr/\sin\phi)\overline{[w_n(z_i)]^2 + [u_{\phi n}(z_i)]^2}\Delta z + (hr/\sin\phi)\overline{[w_n(z_N)]^2 + [u_{\phi n}(z_N)]^2}(\Delta z/2)\right\}} \quad (n = 0) \quad (50)$$

$$\omega^2 = \frac{(4g/\pi)\left\{(hr/\sin\phi)[w_n(z_0) + u_{\phi n}(z_0)](\Delta z/2) + \sum_{i=1}^{N-1}(hr/\sin\phi)[w_n(z_i) + u_{\phi n}(z_i)]\Delta z + (hr/\sin\phi)[w_n(z_N) + u_{\phi n}(z_N)](\Delta z/2)\right\}}{\left\{(hr/\sin\phi)\overline{[w_n(z_0)]^2 + [u_{\phi n}(z_0)]^2 + [u_{\theta n}(z_0)]^2}(\Delta z/2) + \sum_{i=1}^{N-1}(hr/\sin\phi)\overline{[w_n(z_i)]^2 + [u_{\phi n}(z_i)]^2 + [u_{\theta n}(z_i)]^2}\Delta z + (hr/\sin\phi)\overline{[w_n(z_N)]^2 + [u_{\phi n}(z_N)]^2 + [u_{\theta n}(z_n)]^2}(\Delta z/2)\right\}} \quad (n \geq 1) \quad (51)$$

With the circular frequency ω determined from one of the Eqs. (50) and (51), we choose an increment Δt for the solution of our system of equations to be

$$\Delta t = 2\mu\pi/\omega \quad (52)$$

where μ is a parameter that we are free to select for each particular problem.

Results and Conclusions

The equations for the analysis of our shell were programed in FORTRAN IV language and all typical solutions were obtained on the IBM 7094 computer. This necessarily restricted the size of the shell structures which were analyzed, and in actual applications a larger computer will be required for the analysis.

As an illustration of the results obtained from the solution of our system of equations for typical shells we include here the analysis of a cylindrical shell with the geometry and loading shown in Fig. 4. We assume that the initial displacements and velocities are zero. For the boundary conditions we assume that w, u_ϕ, M_ϕ, and u_θ are zero at z_0 and that w, N_ϕ, M_ϕ, and u_θ are zero at z_N. We assume a value of 30×10^6 lb/in.^{2} for E, a value of 0.284 lb/in.^{3} for γ, and a value of 0.30 for ν.

Under the given loading and imposed boundary conditions, all Fourier components designated with a bar are zero, and only the equations containing the variables and loading terms designated without a bar enter into the solution. We obtain for illustrative purposes a sufficiently good representation of the loading by using only the Fourier components for $n = 0$ through $n = 4$. Of these components, the one for $n = 3$ is zero. Thus, the four nonzero components entering into the solution are $p_0 = -318.0$, $p_1 = -500.0$, $p_2 = -212.0$, and $p_4 = 42.0$ lb/in.^{2}. We choose an increment Δz equal to h, the thickness of the shell, thus dividing the length of the cylinder into 36 increments.

We obtain a value of 30,805 rad/sec for the highest fundamental frequency ω by using $n = 0$ in Eq. (50). The fundamental frequencies for the other values of n are found from Eq. (51) to be 24,625 rad/sec for $n = 1$; 19,196 rad/sec for $n = 2$; and 11,684 rad/sec for $n = 4$. The periods corresponding to these frequencies are 20.4×10^{-5} sec for $n = 0$; 25.5×10^{-5} sec for $n = 1$; 32.7×10^{-5} sec for $n = 2$; and 53.7×10^{-5} sec for $n = 4$. With the chosen value for Δz, we

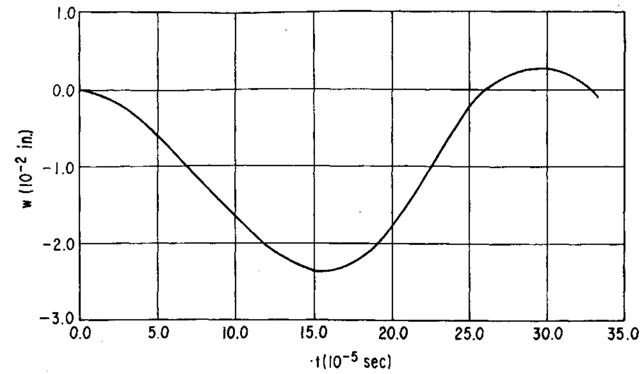

Fig. 5 Plot of $w(z_{18}, t)$ at $\theta = 0$ for illustrative example.

find solutions to be unstable for chosen values of $\Delta t \geqq 0.20 \times 10^{-5}$ sec. We find solutions which are in agreement and stable for chosen values of 0.15×10^{-5}, 0.125×10^{-5}, and 0.0625×10^{-5} sec for Δt. No solutions were attempted for values of $\Delta t < 0.0625 \times 10^{-5}$ sec. With the chosen Δz and either of the three latter choices of Δt we obtain, after solving our system of equations for each of the four Fourier components of loading, values of the fundamental variables at the selected points of interest on the shell by use of Eqs. (11–18). Values of the stress resultants other than the fundamental variables are obtained from Eqs. (19–24).

We illustrate the results of our solution through the half fundamental periods for all participating Fourier components by showing for the meridian $\theta = 0$ values of $w(z_{18},t)$ obtained by using both $\Delta t = 0.125 \times 10^{-5}$ sec and $\Delta t = 0.150 \times 10^{-5}$ sec in Table 1. For comparison purposes we find the value of this function under a static application of the loading shown in Fig. 4 to be -1.0647×10^{-2} in for the meridian $\theta = 0$. We obtain this static solution by setting Δt equal to infinity and solving the equations for the first time step. It is seen from the results shown in Table 1 that the solutions obtained by use of $\Delta t = 0.125 \times 10^{-5}$ sec and $\Delta t = 0.150 \times 10^{-5}$ sec are in agreement. To illustrate graphically the nature of the shell response we show through the half fundamental periods for all participating Fourier components, for the meridian $\theta = 0$, plots of $w(z_{18},t)$ and $Q(z_0,t)$ in Figs. 5 and 6, respectively.

The results shown in Table 1 and in Figs. 5 and 6 for the typical example indicate that very good solutions for the response of rotationally symmetric shells under time dependent loadings and boundary conditions may be obtained by the finite difference representations which have been used. The determination of an appropriate Δt to be used with a given Δz constitutes the only difficulty in obtaining stable solutions for given shell geometries and loadings. For the illustrative example a Δt for which the solutions were stable was found with the fourth trial value. After solving a few examples with the program it should be possible to establish a range of values of the parameter μ for which solutions are usually found to be stable. Thus, it is expected that a very good value for Δt can then be chosen initially. It should also be noted that any instabilities usually manifest themselves in the results obtained after not more than five or six time steps. Thus, no lengthy computations need to be made with an unsuitable Δt. Additionally, for the second choice of Δt, the solutions should be made for perhaps not more than twenty-five time steps to verify the stability of the solutions.

The results obtained for typical examples with different choices of Δt agree generally to three, four, and five significant figures for the several dependent variables. This accuracy may be increased as much as desired by increasing the order of the finite difference representations, particularly on the boundaries of the shell, or by suitably reducing the size of the increment Δz. By suitably reducing the sizes of the increments Δz and Δt, we may also take into account the exciting of

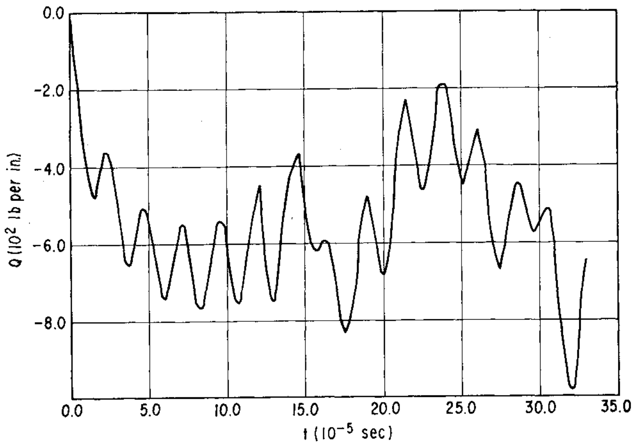

Fig. 6 Plot of $Q(z_0, t)$ at $\theta = 0$ for illustrative example.

vibration modes of still shorter wave lengths and higher frequencies. The solutions can also be refined by considering inertia forces in the circumferential direction of the shell in addition to the inertia forces transverse to the middle surface and along the meridian of the shell already considered. This would yield an additional explicit expression for $u_{\theta n}(z,t)$, and the number of equations to be solved for the remaining unknown quantities at any time increment would be correspondingly reduced.

With the program which has been written, all equations are solved to single precision. In the process of checking the solutions obtained for the illustrative example it was found that results obtained by using double precision in the calculations agreed with the results found by using single precision. Thus, it is expected that single precision calculations will be sufficient for most shell problems. We conclude that the finite difference methods employed here constitute a most expeditious procedure for the dynamic analysis of shells.

References

[1] Penny, R. K., "Symmetric Bending of the General Shell of Revolution by Finite Difference Methods," *Journal of Mechanical Engineering Science*, Vol. 3, No. 4, Dec. 1961, pp. 369–377.

[2] Radkowski, P. P., Davis, R. M., and Bolduc, M. R., "Numerical Analysis of Equations of Thin Shells of Revolution," *ARS Journal*, Vol. 32, No. 1, Jan. 1962, pp. 36–41.

[3] Budiansky, B. and Radkowski, P. P., "Numerical Analysis of Unsymmetrical Bending of Shells of Revolution," *AIAA Journal*, Vol. 1, No. 8, Aug. 1963, pp. 1833–1842.

[4] Kalnins, A., "Analysis of Shells of Revolution Subjected to Symmetrical and Nonsymmetrical Loads," *Transactions ASME, Ser., 31 E: Journal of Applied Mechanics*, Vol. 3, Sept. 1964, pp. 467–476.

[5] Percy, J. H. et al., "Application of Matrix Displacement Method to Linear Elastic Analysis of Shells of Revolution," *AIAA Journal*, Vol. 3, No. 11, Nov. 1965, pp. 2138–2145.

[6] Kalnins, A., "Free Vibration of Rotationally Symmetric Shells," *Journal of Acoustical Society of America*, Vol. 36, July 1964, pp. 1355–1365.

[7] Kraus, H. and Kalnins, A., "Transient Vibration of Thin Elastic Shells," *Journal of Acoustical Society of America*, Vol. 38, No. 6, Dec. 1965, pp. 994–1002.

[8] Klein, S., "Vibrations of Multi-Layer Shells of Revolution Under Dynamic and Impulsive Loading," *Shock and Vibration Bulletin*, Vol. 35, No. 3, Jan. 1966, pp. 27–44.

[9] Johnson, D. E. and Greif, R., "Dynamic Response of a Cylindrical Shell: Two Numerical Methods," *AIAA Journal*, Vol. 4, No. 3, March 1966, pp. 486–494.

[10] Reissner, E., "A New Derivation of the Equations for the Deformation of Elastic Shells," *American Journal of Mathematics*, Vol. 63, 1941, pp. 177–184.

[11] Smith, T. A., "Numerical Solution for the Dynamic Response of Rotationally Symmetric Shells of Revolution Under Transient Loadings," TR RS-TR-70-5, June 1970, U.S. Army Missile Command, Redstone Arsenal, Ala.